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N-loop solitons and their link with the complex Harry Dym equation

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Abstract. N-loop solitons are constructed by means of N-cusp soliton solutions of the Harry Dym equation. The proposed approach is based on a novel link between the 'loop soliton equation' and the complex Harry Dym equation considered on a special curve.

1. Introduction

This paper deals with constructing the multisoliton solutions of the nonlinear evolution equation

$$y_{\xi t} + \operatorname{sgn}\left(\frac{\mathrm{d}\xi}{\mathrm{d}s}\right)\left(\frac{y_{\xi\xi}}{(1+y_\xi^2)^{3/2}}\right)_{\xi\xi} = 0 \tag{1}$$

where $y = y(\xi, t)$, ξ and $t \in \mathbb{R}$, $y \to 0$ as $\xi \to \pm\infty$ and s denotes the arc length measured at any fixed t along the plot of the function $y = y(\xi, t)$ up to the point under consideration.

From the physical point of view the nonlinear system (1) arises [1] as the generalization of the equation which describes the nonlinear transverse oscillation of elastic beams under tension. This equation has the form

$$y_{\xi t} + \left(\frac{y_{\xi\xi}}{(1+y_\xi^2)^{3/2}}\right)_{\xi\xi} = 0 \tag{2}$$

and can be obtained [2] by a reduction of the rigorous equation of motion of a uniform elastic beam under tension derived in [3]. The latter reads as follows:

$$u_{TT} - \lambda^2 u_{XX} + \frac{EI}{\rho A}\partial_X^2\left[\frac{u_{XX}}{(1+u_X)^{3/2}}\right] = 0 \tag{3}$$

where X and u are the coordinates of the plane in which the beam is situated, T is the time variable, $\lambda = (\mu/\rho A)^{1/2}$ is the linear wave velocity, ρ is the density of the material, A denotes the area of cross section, and E and I are Young's modulus and the moment of inertia, respectively. Finally, μ is the tension parallel to the axis of X and is assumed to be constant.

Equation (3) can be reduced to equation (2) under the change of variables $\xi = A^{-1/2}(X + \lambda T)$, $y = A^{-1/2}u$, $t = \varepsilon A^{-1/2}\lambda T$ by keeping up to the first order of the dimensionless parameter $\varepsilon = EI/(2\mu A)$. It measures the relative size of the bending stiffness under tension along the beam and is assumed to be small. It should be noted that the nonlinear term of equation (3) (and consequently of equation (2)) is due to the bending moment, which is proportional to the curvature of the beam $\kappa = u_{XX}(1+u_X^2)^{3/2}$.

Now instead of a beam under tension let us consider a flexible elastic stretched rope on which the deformation takes the shape of a loop. One can describe this situation just as in the case of small deformations of the rigid beam, but now it is necessary to distinguish the branches of the deformation of the stretched rope with different signs of curvature. With this aim the expression of the curvature has to be modified in the following way [1]:

$$\kappa \equiv \frac{\mathrm{d}\Theta}{\mathrm{d}s} = \frac{\mathrm{d}X}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}s}\left(\tan^{-1} u_X\right) = \mathrm{sgn}\left(\frac{\mathrm{d}X}{\mathrm{d}s}\right)\left(1+u_X^2\right)^{-3/2} u_{XX}$$

where $\mathrm{d}\Theta$ is an increment of the tangential angle Θ at a point of the stretched rope, and $\mathrm{d}s$ is an increment of the arc length. The factor $\mathrm{sgn}(\mathrm{d}X/\mathrm{d}s)$ is regarded as an index to define the branches of deformation having opposite sign to that of the curvature. The consequence of the above definition of curvature is the appearance of this sign factor in the nonlinear term of equation (3). Now applying to equation (3), modified as above, the same transformations and assumptions which reduced equation (3) to equation (2), one easily obtains equation (2) with the factor $\mathrm{sgn}(\mathrm{d}\xi/\mathrm{d}s)$ in the nonlinear term, i.e. equation (1). Thus the latter describes a wave propagating along a stretched rope when the parameter measuring the relative size of the bending stiffness under tension along the rope is considered to be small.

Equation (1), as well as equation (2), both belong to a class of integrable nonlinear equations found by Wadati, Konno and Ichikawa [4] (hereafter referred to as WKI equations). Among WKI equations one can also mention the Harry Dym equation [5] and the nonlinear Schrödinger-type equation [6].

The distinctive feature of WKI equations is that the associated linear problem differs from the Zakharov–Shabat spectral problem that generates the 'usual' AKNS equations [7]. The crucial point of the WKI spectral problem is the dependence of the wavefunction's asymptotics (as the spectral parameter tends to infinity) on the coefficients of the system. As a consequence, all known solutions of the WKI equations are intrinsically implicit. This means that they depend on space and time variables through some phase function, the latter being defined implicitly by a functional equation [1, 6, 8–11]. That is why, in spite of the fact that each WKI equation may be linked by a sequence of transformations [12, 13] with an AKNS equation, WKI equations possess many unusual features, and indeed form a new class of integrable equations deserving special study.

The one-soliton and two-soliton solutions of equation (1) were found by the inverse scattering transform (IST) in [1, 9]. They turned out to have the shape of loops and be described as follows:

$$y(\xi, t) = Y(\xi + \varepsilon_-(\xi, t), t) \tag{4}$$

where the function $\varepsilon_-(\xi, t)$ is defined implicitly by the functional equation

$$\varepsilon_-(\xi, t) = E_-(\xi + \varepsilon_-(\xi, t), t)\,. \tag{5}$$

Here Y and E_- are some functions discussed below.

The one-loop soliton solution of equation (1) was also obtained in [14] due to the established transformation with the modified Korteweg–de Vries (mKdV) equation. This transformation was applied for the case of N-loop solitons in [13]. However, it gave only a parametric representation for such solutions. Namely, the functions Y and E_- in equations (4), (5) were written down as integrals of some known functions. These integrals can only easily be calculated analytically in the one-soliton case. Even for $N = 2$ one meets serious analytical difficulties, to say nothing of those arising for arbitrary N.

An attempt to find the functions Y and E_- explicitly when $N \geqslant 3$ by the IST fails due to even greater analytical difficulties.

In the present paper we propose a method which allows us to find the above functions explicitly for arbitrary N. Thus we construct by nature implicit N-loop soliton solutions of equation (1) (it will hereafter be called the 'loop soliton equation' (LSE)) in 'almost explicit' form. For $N = 1, 2$ the results obtained coincide with the known ones [1, 9, 14].

Our approach is based on a link which we establish between the LSE and the complex Harry Dym equation (cHDE)

$$\hat{r}_t + \hat{r}^3\hat{r}_{zzz} = 0 \qquad \hat{r} = \hat{r}(z, t) \in \mathbb{C} \tag{6}$$

where the complex variable $z = \xi + \mathrm{i}y(\xi, t)$ at any fixed $t \in \mathbb{R}$ changes along the plot of the LSE solution $y(\xi, t)$ in the plane (ξ, y). The $2N$-parametric complex solutions of equation (6) are obtained by a simple procedure from the N-cusp soliton solutions of the 'real' HDE. The latter have recently been constructed in [10] by the 'higher times approach'. Finally, the N-loop soliton solutions of the LSE are derived by means of the above-mentioned $2N$-parametric solutions of the cHDE.

2. N-loop soliton solutions

The loop soliton equation (1) is known [14] to be linked with the mKdV equation in potential form

$$\Theta_\tau + \tfrac{1}{2}\Theta_\chi^3 + \Theta_{\chi\chi\chi} = 0 \qquad \Theta = \Theta(\chi, \tau) \tag{7}$$

considered under the boundary conditions $\Theta \to 2\pi$ as $\chi \to +\infty$ and $\Theta \to 0$ as $\chi \to -\infty$. This link is based on the change of variables

$$\chi = \xi + \varepsilon_-(\xi, t) \qquad \tau = t \tag{8}$$

where the phase function $\varepsilon_-(\xi, t)$ is defined implicitly in terms of the mKdV solution Θ by the functional equation

$$\varepsilon_-(\xi, t) = E_-(\chi, \tau) \equiv \int_{-\infty}^{\chi} (1 - \cos\Theta)\,\mathrm{d}\chi' . \tag{9}$$

The solution $y(\xi, t)$ of equation (1) is then given as follows:

$$y(\xi, t) = Y(\chi, \tau) \equiv \int_{-\infty}^{\chi} \sin\Theta\,\mathrm{d}\chi' . \tag{10}$$

Equations (9), (10) yield a parametric representation for an LSE solution in terms of some solution of the mKdV (7). In [13] it was shown that the mKdV N-soliton solution [7] generates the N-loop soliton solution of the LSE in such a way. The first of them can be written in the form

$$\Theta(\chi, \tau) = 4\tan^{-1}\frac{\Phi_u}{\Phi_g} \tag{11}$$

where

$$\Phi_u(\chi, \tau) = \sum_{m_u \in \mathbb{Z}^N}{}' (-1)^{(s-1)/2} \exp\left(\tfrac{1}{2}\langle Bm_u, m_u\rangle + \langle \eta, m_u\rangle\right) \tag{12}$$

$$\Phi_g(\chi, \tau) = \sum_{m_g \in \mathbb{Z}^N}{}' (-1)^{s/2} \exp\left(\tfrac{1}{2}\langle Bm_g, m_g\rangle + \langle \eta, m_g\rangle\right) . \tag{13}$$

Here $\sum'$ denotes the summation over the vectors m_u or $m_g \in \mathbb{Z}^N$ whose components are equal only to 0 or 1. The number s of components equal to 1 is odd for the vector m_u: $s = 2k+1$, $k = 0, 1, \ldots, \left[\frac{N-1}{2}\right]$. For the vector m_g this number is even: $s = 2k$, $k = 0, 1, \ldots, \left[\frac{N}{2}\right]$. The vector $\eta \equiv \eta(\chi, \tau) = (\eta_1, \ldots, \eta_N)$ is given by

$$\eta_k = p_k\chi - p_k^3\tau - \eta_k^0 \qquad k = 1, \ldots, N \tag{14}$$

with p_k ($p_l \neq p_j$ as $l \neq j$) and η_k^0 being arbitrary real constants. By $\langle\cdot,\cdot\rangle$ we denote the standard inner product $\langle\eta, m\rangle = \sum_{k=1}^N \eta_k m_k$. The symmetric $N \times N$ matrix B has the entries

$$B_{kl} = 2\ln\left|\frac{p_k - p_l}{p_k + p_l}\right| \qquad B_{kk} = 0\,. \tag{15}$$

One can express $\cos\Theta$ and $\sin\Theta$ in (9), (10) in terms of Φ_u and Φ_g and thus obtain the parametric representation for the N-loop soliton solution given in [13]. However, due to the complicated dependence of Φ_u and Φ_g on χ an explicit calculation of integrals on the right-hand side of (9), (10) for arbitrary N is an intractable analytical problem.

To avoid this problem, we now show how to find the functions $E_-(\chi, \tau)$ and $Y(\chi, \tau)$ in (9), (10) explicitly by using our previous results [10] on the N-cusp soliton solutions of the Harry Dym equation (HDE)

$$r_t + r^3 r_{xxx} = 0 \qquad r = r(x, t) \qquad x, t \in \mathbb{R}\,. \tag{16}$$

These solutions being intrinsically implicit are described on 'almost explicit' level in the form

$$r(x, t) = R(\chi, \tau)$$

where

$$\chi = x + \varepsilon(x, t) \qquad t = \tau$$

with

$$\varepsilon(x, t) = E(\chi, \tau)\,.$$

The functions R and E are given explicitly by expressions

$$R(\chi, \tau) = \left[\frac{\sum'_{m\in\mathbb{Z}^N} \exp(\frac{1}{2}\langle Bm, m\rangle + \langle\eta + \mathrm{i}\pi m, m\rangle)}{\sum'_{m\in\mathbb{Z}^N} \exp(\frac{1}{2}\langle Bm, m\rangle + \langle\eta, m\rangle)}\right]^2 \tag{17}$$

$$E(\chi, \tau) = 4\frac{\sum'_{m\in\mathbb{Z}^N} \exp(\frac{1}{2}\langle Bm, m\rangle + \ln\langle\beta, m\rangle + \langle\eta, m\rangle)}{\sum'_{m\in\mathbb{Z}^N} \exp(\frac{1}{2}\langle Bm, m\rangle + \langle\eta, m\rangle)} \tag{18}$$

and are related as follows:

$$E(\chi, \tau) = \int_{-\infty}^{\chi} [1 - R(\chi', \tau)]\,\mathrm{d}\chi'. \tag{19}$$

In equations (17), (18) $\sum'$ denotes, as earlier, the summation over the vector $m \in \mathbb{Z}^N$, whose components are equal to 0 or 1. The vector $\eta \equiv \eta(\chi, \tau) = (\eta_1, \ldots, \eta_N)$ is given by (14), the matrix B is defined as in (15) and the components of the vector $\beta \in \mathbb{R}^n$ read

$$\beta_k = p_k^{-1}\,. \tag{20}$$

Now we return to equations (8)–(10). They can be written in the form

$$y(\xi, t) = Y(\chi, \tau) \equiv -\operatorname{Im} \hat{E}(\chi, \tau) \tag{21}$$

$$\chi = \xi + \varepsilon_-(\xi, t) \qquad \tau = t$$

$$\varepsilon_-(\xi, t) = E_-(\chi, \tau) \equiv \operatorname{Re} \hat{E}(\chi, \tau) \tag{22}$$

where

$$\hat{E}(\chi, \tau) = \int_{-\infty}^{\chi} [1 - \hat{R}(\chi', \tau)]\, \mathrm{d}\chi' \tag{23}$$

with

$$\hat{R}(\chi, \tau) \equiv \exp[\mathrm{i}\Theta(\chi, \tau)]\,. \tag{24}$$

Since $\hat{R}(\chi, \tau)$ and, consequently, $\hat{E}(\chi, \tau)$ depends on χ and τ by means of the vector η (14), it is natural to set

$$\hat{R}(\chi, \tau) \equiv \hat{R}(\eta_1, \ldots, \eta_N) \qquad \hat{E}(\chi, \tau) \equiv \hat{E}(\eta_1, \ldots, \eta_N)\,.$$

We introduce analogous designations for the functions $R(\chi, \tau)$ and $E(\chi, \tau)$ given by (17), (18). The links between the functions $\hat{R}$ and R, as well as $\hat{E}$ and E, are stated by the following lemma.

Lemma.

$$\hat{R}(\eta_1, \ldots, \eta_N) = R\left(\eta_1 - \frac{\mathrm{i}\pi}{2}, \ldots, \eta_N - \frac{\mathrm{i}\pi}{2}\right) \tag{25}$$

$$\hat{E}(\eta_1, \ldots, \eta_N) = E\left(\eta_1 - \frac{\mathrm{i}\pi}{2}, \ldots, \eta_N - \frac{\mathrm{i}\pi}{2}\right). \tag{26}$$

Proof. Due to equations (24) and (11) the function $\hat{R}$ can be written as

$$\hat{R} = \left[\frac{\Phi_g + \mathrm{i}\Phi_u}{\Phi_g - \mathrm{i}\Phi_u}\right]^2. \tag{27}$$

Then we rewrite the function R given by (17) in the form

$$R = \left[\frac{f_g - f_u}{f_g + f_u}\right]^2 \tag{28}$$

where

$$f_{u(g)}(\chi, \tau) = \sum_{m_{u(g)} \in \mathbb{Z}^N}{}' \exp\left(\tfrac{1}{2}\langle B m_{u(g)}, m_{u(g)}\rangle + \langle \eta, m_{u(g)}\rangle\right).$$

Here all the designations are the same as in (12), (13). Comparing the latter with expressions for $f_{u(g)}(\chi, \tau) \equiv f_{u(g)}(\eta_1, \ldots, \eta_N)$ one can see that

$$\Phi_g(\eta_1, \ldots, \eta_N) = f_g\left(\eta_1 - \frac{\mathrm{i}\pi}{2}, \ldots, \eta_N - \frac{\mathrm{i}\pi}{2}\right)$$

$$\Phi_u(\eta_1, \ldots, \eta_N) = \mathrm{i} f_u\left(\eta_1 - \frac{\mathrm{i}\pi}{2}, \ldots, \eta_N - \frac{\mathrm{i}\pi}{2}\right).$$

The last relations with equations (27), (28) give the link (25). Equation (26) now follows from it due to transformations (19) and (23). □

With the use of the explicit representation (18) for E the above-proved lemma allows us to obtain the following representation for the function $\hat{E}$:

$$\hat{E} = 4\frac{\Psi_g - \mathrm{i}\Psi_u}{\Phi_g - \mathrm{i}\Phi_u} \tag{29}$$

where the functions $\Phi_{u(g)}$ are given by (12), (13), and the functions $\Psi_{u(g)}$ read as

$$\Psi_u(\chi, \tau) = \sum_{m_u \in \mathbb{Z}^N}{}' (-1)^{(s-1)/2} \exp\left(\tfrac{1}{2}\langle Bm_u, m_u\rangle + \ln\langle\beta, m_u\rangle + \langle\eta, m_u\rangle\right) \tag{30}$$

$$\Psi_g(\chi, \tau) = \sum_{m_g \in \mathbb{Z}^N}{}' (-1)^{s/2} \exp\left(\tfrac{1}{2}\langle Bm_g, m_g\rangle + \ln\langle\beta, m_u\rangle + \langle\eta, m_g\rangle\right). \tag{31}$$

Here again we use designations similar to those in equations (12), (13). The vector β is given by (20).

Inserting the last equations into (29), (21), (22) we obtain the N-loop soliton solution on 'almost explicit' level. We formulate the main result of the present paper as follows.

Theorem 1. The N-loop soliton solutions of the LSE (1) have the form

$$y(\xi, t) = Y(\chi, \tau) \equiv 4\frac{\Psi_u\Phi_g - \Psi_g\Phi_u}{\Phi_g^2 + \Phi_u^2} \tag{32}$$

where

$$\chi = \xi + \varepsilon_-(\xi, t) \qquad \tau = t \tag{33}$$

with

$$\varepsilon_-(\xi, t) = E_-(\chi, \tau) \equiv 4\frac{\Psi_g\Phi_g + \Psi_u\Phi_u}{\Phi_g^2 + \Phi_u^2} \tag{34}$$

Here $\Phi_{u(g)}$ and $\Psi_{u(g)}$ are given by equations (12), (13), (30), (31) through the use of equations (14), (15), (20).

3. A link with the complex Harry Dym equation

In this section we show a link between the LSE (1), mKdV equation (7) and cHDE (6) considered on a special curve. This link clarifies the intrinsic nature of the constructions given in section 2.

Suppose that $\Theta(\chi, \tau)$ is a solution of the mKdV equation (7) with boundary conditions $\Theta \to 2\pi$ as $\chi \to +\infty$, $\Theta \to 0$ as $\chi \to -\infty$. The LSE solution $y(\xi, t)$ ($y \to 0$ as $\xi \to \pm\infty$) is linked with Θ by the Ishimori transformation (8)–(10). We also introduce the complex function

$$\hat{r}(z, t) = \hat{R}(\chi, \tau) \equiv \exp[\mathrm{i}\Theta(\chi, \tau)] \tag{35}$$

where $z \in \mathbb{C}$, with

$$z = -\hat{E}(\chi, \tau) + \chi \equiv \int_{-\infty}^{\chi} [\hat{R}(\chi', \tau) - 1]\,\mathrm{d}\chi' + \chi \qquad t = \tau. \tag{36}$$

Theorem 2. The function $\hat{r}(z,t)$ satisfies the cHDE (6) considered in the complex plane $z=\xi+\mathrm{i}y$ on the above-mentioned LSE solution $y(\xi,t)$ plot against ξ.

Proof. To prove this statement we use the known links [12, 15] between the 'real' HDE (16) and the mKdV in the form

$$\varphi_\tau - \tfrac{1}{2}\varphi_\chi^3 + \varphi_{\chi\chi\chi} = 0 . \tag{37}$$

These links are

$$r(x,t) = R(\chi,\tau) \equiv \exp\varphi(\chi,\tau) \tag{38}$$

$$x = \int_{-\infty}^{\chi} [R(\chi',\tau)-1]\,\mathrm{d}\chi' + \chi \qquad t=\tau . \tag{39}$$

On using (38), equation (37) yields

$$R_\tau + \tfrac{3}{2}R_\chi^3 R^{-2} - 3R_{\chi\chi}R_\chi R^{-1} + R_{\chi\chi\chi} = 0 . \tag{40}$$

The latter equation under the change of variables (39) transforms into the HDE $r_t + r^3 r_{xxx} = 0$.

Now one can easily check that if the function $\Theta(\chi,\tau)$ solves the mKdV equation (7) the *function $\hat{R}$ defined by (35) satisfies the same equation (40) as does R. Since the reciprocal* transformations (36) and (39) are similar, the function $\hat{r}(z,t)$ satisfies the cHDE (6).

To show now that at any fixed t the variable $z=\xi+\mathrm{i}y$ changes along the LSE solution $y(\xi,t)$ plot we make use of equations (8)–(10) which define this solution. They can be written as

$$\xi = \int_{-\infty}^{\chi} (\cos\Theta - 1)\,\mathrm{d}\chi' + \chi \qquad y(\xi,t) = \int_{-\infty}^{\chi} \sin\Theta\,\mathrm{d}\chi' .$$

Comparing the last equations with (36), (35) one can see that

$$z = \xi + \mathrm{i}y(\xi,t) .$$

Thus, theorem 2 is proved. □

Remark. From equations (8)–(10) it follows that $y_\xi = \tan\Theta$. Thus the real and imaginary parts of the cHDE solution $\hat{r} = \cos\Theta + \mathrm{i}\sin\Theta$ at any ξ form components of the tangent vector to the plot of $y(\xi,t)$ (t is fixed) against ξ.

In section 2 the mKdV solution Θ was chosen as the N-soliton solution (11)–(13). In this case the functions $\hat{R}$ and $\hat{E}$ which completely define the above-discussed cHDE solution $\hat{r}$ (see (35), (36)) are given explicitly by (27) and (29), respectively. This $2N$-parametric complex solution was obtained from the known [10] N-cusp soliton solution of the HDE by means of the transformations (25), (26). Finally, the LSE N-loop soliton solution itself was constructed due to the links (21), (22).

4. Particular cases of the N-loop solitons: N = 1, 2, 3

For $N=1$, equations (12), (13), (30), (31) yield

$$\Phi_u = \exp\eta_1 \qquad \Phi_g = 1$$

$$\Psi_u = p_1^{-1}\exp\eta_1 \qquad \Psi_g = 0 .$$

Then equations (32), (34) take the form

$$y(\xi, t) = 2p_1^{-1}\operatorname{sech}\eta_1$$
$$\varepsilon_-(\xi, t) = 2p_1^{-1}\exp\eta_1\operatorname{sech}\eta_1 \qquad (41)$$

where

$$\eta_k = p_k(\xi + \varepsilon_-(\xi, t)) - p_k^3 t - \eta_k^0 \qquad k = 1\,. \qquad (42)$$

The function ε_- satisfies the boundary conditions: $\varepsilon_- \to 0$ as $\xi \to -\infty$ and $\varepsilon_- \to 4p_1^{-1}$ as $\xi \to +\infty$.

We note that in [1, 14] concerning the one-loop soliton another phase function $\varepsilon_+(\xi, t)$ was used. It was introduced under the boundary conditions $\varepsilon_+ \to 0$ as $\xi \to +\infty$ and is related to our phase function as follows:

$$\varepsilon_+ = \varepsilon_- - \frac{4}{p_1}\,.$$

Hence

$$\varepsilon_+(\xi, t) = \frac{2}{p_1}(\tanh\eta_1 - 1)\,. \qquad (43)$$

Equations (41), (43) coincide with the known LSE one-loop soliton solution.

For $N = 2$ one has

$$\Phi_u = \exp\eta_1 + \exp\eta_2 \qquad (44)$$

$$\Phi_g = 1 - \gamma_{12}\exp(\eta_1 + \eta_2) \qquad (45)$$

$$\Psi_u = p_1^{-1}\exp\eta_1 + p_2^{-1}\exp\eta_2 \qquad (46)$$

$$\Psi_g = -\gamma_{12}(p_1^{-1} + p_2^{-1})\exp(\eta_1 + \eta_2) \qquad (47)$$

where $\eta_k (k = 1, 2)$ is given by (42) to be

$$\gamma_{jk} = \left(\frac{p_j - p_k}{p_j + p_k}\right)^2 \qquad j, k = 1, 2. \qquad (48)$$

An insertion of the functions (44)–(47) into (32), (34) leads to the known results [9] on 2-loop soliton solutions (to obtain a complete correspondence one has to take into account the substitution $\xi \leftrightarrow -\xi$, $\varepsilon_+ \leftrightarrow -\varepsilon_+$, where $\varepsilon_+ = \varepsilon_- - 4(p_1^{-1} + p_2^{-1})$, which is to be implied in comparison with [9]).

Finally, for $N = 3$, equations (12), (13), (30), (31) take the form

$$\Phi_u = \exp\eta_1 + \exp\eta_2 + \exp\eta_3 - \gamma_{12}\gamma_{13}\gamma_{23}\exp(\eta_1 + \eta_2 + \eta_3)$$

$$\Phi_g = 1 - \gamma_{12}\exp(\eta_1 + \eta_2) - \gamma_{13}\exp(\eta_1 + \eta_3) - \gamma_{23}\exp(\eta_2 + \eta_3)$$

$$\Psi_u = p_1^{-1}\exp\eta_1 + p_2^{-1}\exp\eta_2 + p_3^{-1}\exp\eta_3$$
$$-\gamma_{12}\gamma_{13}\gamma_{23}(p_1^{-1} + p_2^{-1} + p_3^{-1})\exp(\eta_1 + \eta_2 + \eta_3)$$

$$\Psi_g = -\gamma_{12}(p_1^{-1} + p_2^{-1})\exp(\eta_1 + \eta_2) - \gamma_{13}(p_1^{-1} + p_3^{-1})\exp(\eta_1 + \eta_3)$$
$$- \gamma_{23}(p_2^{-1} + p_3^{-1})\exp(\eta_2 + \eta_3)$$

where $\eta_k (k = 1, 2, 3)$ are given by (42), and the constants $\gamma_{jk} (j, k = 1, 2, 3)$ have the form (48). The above functions, when inserted into equations (32), (34) describe the 3-loop soliton solution on an 'almost explicit' level.

In conclusion, we notice that the method proposed in the present paper can be generalized to the case of periodic boundary conditions for the LSE. The corresponding LSE solutions can be obtained by means of the finite-gap solutions of the HDE which have been recently constructed in [11].

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